Combining Algebraic Effect Descriptions using the Tensor of Complete Lattices

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Abstract

Algebras can be used to interpret the behaviour of effectful programs. In particular, we use Eilenberg-Moore algebras given over a complete lattices of truth values, which specify answers to queries on programs. The algebras can be used to formulate a quantitative logic of behavioural properties, specifying a congruent notion of program equivalence coinciding with a notion of applicative bisimilarity. Many combinations of effects can be interpreted using these algebras. In this paper, we specify a method of generically combining effects and the algebras used to interpret them. At the core of this method is the tensor of complete lattices, which combines the carrier set of the algebras. We show that this tensor preserves complete distributivity of complete lattices. Moreover, the universal properties of this tensor can then be used to properly combine the Eilenberg-Moore algebras. We will apply this method to combine the effects of probability, global store, cost, nondeterminism, and error effects. We will then compare this method of combining effects with the more traditional method of combining equational theories using interaction laws.

Keywords: Algebraic effects, Eilenberg-Moore algebra, Tree monad, Complete lattice, Tensor product, Program equivalence, Quantitative logic, Applicative bisimilarity, Probability, Nondeterminism, Global store.

1 Introduction

Effects can alter the behaviour of functional programs in many ways. In order to interpret the behaviour of effectful programs, we choose a set of answers called a *quantitative truth space*, and a set of questions (theoretical tests on programs) given by algebras. In [31], such truth spaces and algebras are used to formulate a logic of quantitative behavioural properties, which induces a notion of program equivalence. This is a generalisation of earlier work based on *modalities* (Boolean algebras) [28,29].

We study *algebraic effects* in the sense of [25]. Such algebraic effects are given by a signature of effect operations. For each effect we choose a truth space A of answers given by a *complete lattice*, and one or more algebras. Such algebras can be specified by *local functions* over the effect operations (forming an observation algebra). If such functions preserve non-empty suprema, they induce an ω-continuous *Eilenberg-Moore algebra*. Many examples of algebraic effects can be expressed using such algebra, including probability, global store, nondeterminism, error, and cost.

An ω-continuous Eilenberg-Moore algebra specifies a notion of *program equivalence* on functional languages with effects and general recursion. This program equivalence can be formulated as *applicative bisimilarity* [1,6,7], and as logical equivalence via a quantitative logic [31,32]. Moreover, the algebra induces a *compositional equational theory*, which for the various examples coincide with the usual equational theories for algebraic effects formulated in the literature (see e.g. [24,26,27] for such equational theories).

The main contribution of this paper is the development of a uniform method for combining effects described by algebras constructed using local functions over effect operations. This method uses the *tensor product* on

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complete lattices, featured e.g. in [10,17,30,34]. This operation is a symmetric monoidal product which naturally combines the different notions of truth for the different effects. We show that the tensor product preserves *complete distributivity* of complete lattices. Complete distributivity is useful for characterising the structure of the tensor of complete lattices.

We combine algebras by combining their local functions, following an elegant definition in terms of the universal property of the tensor product. Given that the complete lattices we use are completely distributive, this combination of local functions preserves the supremum preservation property mentioned before. As such, the induced algebras are suitable for inducing notions of program equivalence.

We compare this method of combining effects with the more traditional way of combining equational theories, which is done by specifying interaction laws such a commutativity [15,16]. We see that our method of combining effects does not uniformly coincide with either the sum or the tensor of equational theories. Instead, it "chooses" interaction laws appropriate to each combination of effects.

In Section 2, we start some preliminaries on algebraic effects and complete lattices. In Section 3, we formulate how to interpret effects using *effect descriptions*, which allow us to construct algebras. In Section 4 we develop a general theory for combining quantitative truth spaces given by completely distributive lattices using the tensor product. Then in Section 5, we use universal properties of the tensor product to combine functions on complete lattices. In Section 6, we formulate how to combine effect descriptions and their algebras, and apply this method to study several combinations of effects. Lastly, in Section 7, we compare this method with methods for combining equational theories.

2 Preliminaries

We represent effects using *algebraic effect operations* following [25]. Each operation op has an *arity* $ar(\mathsf{op}) \in \mathbb{N}$, which tells how many arguments the operation has. For example, we can consider the operation of nondeterministic choice nor, which has arity 2 and chooses nondeterministically between two possible continuations given by its arguments.

For each effect, or combination of effects, we specify an *effect signature* Σ given by a set of effect operations.

Definition 2.1 A Σ-*effect tree* (henceforth *tree*) over a set X is a possibly infinite tree whose nodes are:

(i) leaf nodes labelled $\langle x \rangle$ for $x \in X$,

(ii) leaf nodes labelled $\bot$, describing divergence,

(iii) and internal nodes op for $\mathsf{op} \in \Sigma$, which has $ar(\mathsf{op})$ many children.

We write $\mathsf{op}\langle t_1, \ldots, t_{ar(\mathsf{op})} \rangle$ for the tree whose root has node op, and whose children are given by $t_1, \ldots, t_{ar(\mathsf{op})}$. If a tree only has finitely many nodes, we call it *finite*. We write $T^{\nu}_{\Sigma}(X)$ for the set of trees over X, and $T^{\mu}_{\Sigma}(X)$ for the subset of $T^{\nu}_{\Sigma}(X)$ containing the finite trees over X.

Given an order on X, we can inductively define an order on $T^{\mu}_{\Sigma}(X)$ according to the following rules:

- For any $x, y \in X$, $x \leq_X y \Rightarrow \langle x \rangle \leq_{T^{\mu}_{\Sigma}(X)} \langle y \rangle$.
- For any $t \in T^{\mu}_{\Sigma}(X)$, $\bot \leq_{T^{\mu}_{\Sigma}(X)} t$.
- For any $\mathsf{op} \in \Sigma$, and $l_1, \ldots, l_{ar(\mathsf{op})}, r_1, \ldots, r_{ar(\mathsf{op})} \in T^{\mu}_{\Sigma}(X)$,
  $(\forall 1 \leq i \leq ar(\mathsf{op}). l_i \leq_{T^{\mu}_{\Sigma}(X)} r_i) \Rightarrow \mathsf{op}\langle l_1, \ldots, l_{ar(\mathsf{op})} \rangle \leq_{T^{\mu}_{\Sigma}(X)} \mathsf{op}\langle r_1, \ldots, r_{ar(\mathsf{op})} \rangle$.

The above definition can be adapted to give a coinductive formulation of an order on $T^{\nu}_{\Sigma}(X)$. Moreover, if X is an ω-cpo (it contains limits of increasing sequences), then $T^{\nu}_{\Sigma}(X)$ is an ω-cpo as well. Moreover, each tree is the limit of a sequence of finite trees.

The operations $T^{\mu}_{\Sigma}(-)$ and $T^{\nu}_{\Sigma}(-)$ give endofunctors in the category of posets. Taking some order preserving map $f : X \to Y$, we can lift it to an order preserving map on trees $T^{\mu}_{\Sigma}(f) : T^{\mu}_{\Sigma}(X) \to T^{\mu}_{\Sigma}(Y)$, where $T^{\mu}_{\Sigma}(f)(t)$ is the result of replacing in t each leaf $\langle x \rangle$ with $\langle f(x) \rangle$, leaving the rest of the tree unchanged. We define the function $T^{\nu}_{\Sigma}(f) : T^{\nu}_{\Sigma}(X) \to T^{\nu}_{\Sigma}(Y)$ in the same way.

There is an alternative formulation of these functors. First, we have the partiality functor $(-)_{\bot}$ on posets which adds a minimal element $\bot$ to its argument. Given an effect signature Σ, we define the signature functor F_{Σ} as $F_{\Sigma}(X) = \sum_{\mathsf{op} \in \Sigma} X^{ar(\mathsf{op})}$. Then we define the functors using smallest and largest fixpoint constructions:

$$T^{\mu}_{\Sigma}(X) = \mu Y.(X + F_{\Sigma} Y)_{\bot}, \qquad T^{\nu}_{\Sigma}(X) = \nu Y.(X + F_{\Sigma} Y)_{\bot} \ .$$

Both $T^{\mu}_{\Sigma}(-)$ and $T^{\nu}_{\Sigma}(-)$ form monads in the category of posets, where the unit η is given by $\eta_X(x) = \langle x \rangle$, and the monad multiplication μ is given by locally replacing each leaf $\eta(t)$ by the subtree t (we "flatten" the input).

We end this first section with a brief note on operational semantics. A *computation* is program that needs to be evaluated, or reduced, in order to produce a result. In general, we consider a computation to either return a *value* or *diverge*, potentially encountering effects along the way. As such, we interpret the operational semantics of a programming language as giving a function: $|-| : \mathit{Computations} \to T^{\nu}_{\Sigma}(\mathit{Values})$. In the absence of recursion, this function could be given by $|-| : \mathit{Computations} \to T^{\mu}_{\Sigma}(\mathit{Values})$.

When we test a property of a program, we often start with a property on the values this program can return. For instance, a property of natural numbers can be "Evenness". A property on values is in general given by a quantitative predicate $P : \mathit{Values} \to A$. This uses a truth space A, like the Booleans, or the real number interval $[0, 1]$ of probabilities. To translate this predicate on values, to programs which return such values, we want to lift the predicate P to a quantitative predicate $P' : \mathit{Computations} \to A$. For this purpose, we specify an algebra $\alpha : T^{\nu}_{\Sigma}(A) \to A$ in order to perform the following composition: $(\alpha \circ T^{\nu}_{\Sigma}(P) \circ |-|) : \mathit{Computations} \to A$.

This composition implements the following intuition. We have a program which, after invoking some effects, produces a value according to the operational semantics $|-|$. We have determined a degree of truth for each potential value the program may return, in the form of a quantitative predicate P. Lastly, we have an algebra α, which interprets how effects encountered during the execution of the program combines the degrees of truths of all possible return values, and the effectful ways we could get to those values, into a singular degree of truth.

Take for instance a program which tosses a coin, and on heads will return an 3, and on tails a 6. We want to determine to what degree this program produces an even number. So we take a predicate P which associates to 3 a 0 probability of being even, and to 6 a 1 probability of being even. Then our algebra combines these two results, with the knowledge that the coin is fair. This allows it to combine the result, and determine that the program is expected to produce an even number half of the time.

In the above example, we use as truth space $A = [0, 1]$, the real number interval representing probabilities that certain properties (like "evenness") are held. However, depending on the effect in question, we may need other notions of truth. For instance, for computations using a global store, truth is conditional on the state of this global store upon initiation of evaluation. In that case, we might use the powerset of states as truth space. In the next subsection we establish what properties A, the truth values for our quantitative predicates, needs to satisfy to be useful, keeping it general enough to accommodate a plethora of examples.

2.1 Complete lattices

To interpret effectful behaviour, we use a poset of *truth values*, potential answers to questions asked of (or tests performed on) programs. Intuitively, if $a < b$, then b represents a truth which holds in "more" evaluations of a program than the truth of a. The notion of "more" depends on the effect in question. It may mean: more likely, or for more initial states.

In order for this space of truth values to be suitable for specifying a congruent notion of program equivalence, it needs to be a complete lattice. This has two theoretical reasons:

- In order to construct enough quantitative predicates for distinguishing between behaviourally different programs, we need to close our collection of predicates under arbitrary suprema and infima.
- To formulate applicative bisimilarity, a relation lifting device called a *relator* is defined using our algebras. To facilitate this process, suprema are used to prove that such relators preserve composition of relations.

We will not go further into these points, see [31,32] for more details.

Given a poset A and a subset $X \subseteq A$, we write $\bigvee X$ for the supremum of X: the unique smallest element of A larger than any element in X. Let $\bigwedge X$ be the infimum of X: the unique largest element of S smaller than all elements in X. These elements may not exist. We distinguish between two notions of completeness.

Definition 2.2 A poset A is a *complete lattice* if for any $X \subseteq A$, $\bigvee X$ exists.
A poset A is an *inhabitant complete lattice* (or icomplete lattice) if for any *non-empty* $X \subseteq A$, $\bigvee X$ exists.

If A is a complete lattice, the infimum $\bigwedge X$ always exists, and is equal to $\bigvee\{x \in A \mid \forall y \in X, x \leq y\}$.

Note that a complete lattice is also icomplete. The difference between the two notions of completeness is whether or not the element $\bigvee \emptyset$, the smallest element of A, exists. The top element, $\bigwedge \emptyset = \bigvee A$, always exists in an icomplete lattice. We denote $\mathbf{F}_A$ for the smallest element of A, and $\mathbf{T}_A$ for the biggest element of A.

We expand the intuition of the complete lattice as a truth space, giving degrees of truth associated to a test or property of a program. For instance we make ask ourselves whether: "the program returns an even number". The element $\mathbf{T}_A$ denotes the fact that the property is always observed, regardless of effects that may occur in the evaluation of the program. This happens for instance if the program terminates with a desirable result, without any interference of effects. The element $\mathbf{F}_A$ denotes the fact the property is never observed. This can for instance happen if the program diverges, or produces an undesirable result, without any interference of effects. Even when effects occur, $\mathbf{T}_A$ and $\mathbf{F}_A$ may be attained. For instance, a randomized algorithm may produce different results but still have a 100% probability of producing an even number.

Examples of complete lattices include: The Booleans containing only true and false. The real number interval $[0,1]$ describing probabilities that a statement is true. The powerset $\mathcal{P}(S)$ of states S describing initial states of a global store for. The natural numbers $\mathbb{N}^{\infty}$ with limit element ∞, describing costs.

There is a functor $!(-)$ from the category of icomplete lattices Icom to the category of complete lattices Com, which adds a smallest element to its input. This functor is left adjoint to the forgetful functor $U : \mathsf{Com} \to \mathsf{Icom}$ (which forgets that its input has a smallest element). Note that both $U\circ!$ and $!\circ U$ give an endofunctor akin to the partiality functor $(-)\bot$ on categories Icom and Com respectively.

Importantly, $T^{\mu}_{\Sigma}(-)$ and $T^{\nu}_{\Sigma}(-)$ are not functors in these categories, since $T^{\mu}_{\Sigma}(X)$ and $T^{\nu}_{\Sigma}(X)$ do not tend to have top elements. For instance, to different effect operators do not have a common upper bound. This is why we will remain firmly inside the category of posets when discussing algebras.

We write $\Pi_{i\in I}A_i$ for the I-indexed product of sets A_i. An element of this space is given in lambda notation, $\lambda i.a_i$, which represents an I-indexed family where for each $i \in I$, $a_i \in A_i$.

Definition 2.3 A function $f : \Pi_{i\in I}A_i \to B$ from a product of icomplete lattices to an icomplete lattice is *ilinear* if for any family of nonempty subsets $\{S_i \subseteq A_i\}_{i\in I}$, $f(\lambda i. \bigvee S_i) = \bigvee\{f(\lambda i.x_i) \mid \forall i.x_i \in S_i\}$.

A function $f : \Pi_{i\in I}A_i \to B$ from a product of complete lattices to a complete lattice is *linear* if for any family of subsets $\{S_i \subseteq A_i\}_{i\in I}$, $f(\lambda i. \bigvee S_i) = \bigvee\{f(\lambda i.x_i) \mid \forall i.x_i \in S_i\}$.

If a function is linear, than it is ilinear. Conversely, an ilinear function $f : \Pi_{i\in I}A_i \to B$ on complete lattices is linear if and only if for any family of elements $\{x_i \in A_i\}_{i\in I}$ such that $\exists i \in I.x_i = \mathbf{F}_{A_i}$, then $f(\lambda i.x_i) = \mathbf{F}_B$.

The reason why we distinguish between linearity and ilinearity is because of a disconnect between the demands on the truth space A, and the demands on how to answer questions using an algebra. Linearity is the inherent property of morphisms in Com which contains our truth spaces, whereas the local functions which will make up our algebras are ilinear, not linear (see the forthcoming examples in Subsection 3.2).

A complete lattice is *completely distributive* if the infima distributes over the suprema operation. See for instance [8] for an overview of properties of complete lattices. There is an equivalent definition to complete distributivity using linearity:

Definition 2.4 A complete lattice A is *completely distributive* if for any set I, the infima operation $\bigwedge_I \Pi_{i\in I}A \to A$, sending $\lambda i.x_i$ to $\bigwedge_{i\in I} x_i$, is linear.

Observe that it is sufficient to require the infima operations $\bigwedge_I$ to be ilinear, since they satisfy the extra property of bottom preservation discussed before. Assuming that our truth spaces are completely distributive will be necessary later on for establishing some properties. Note however, that it is always possible to freely generate a completely distributive lattice from any complete lattice. So in theory, this restriction to complete distributivity is not restrictive in terms of applications to describing effects.

3 Effect Descriptions

For each effect, or combination of effects, we choose a completely distributive lattice A of truth values, or space of observables. To interpret the behaviour of effects, we specify an algebra $\alpha : T^{\mu}_{\Sigma}(A) \to A$ in the category of posets. This algebra is constructed using the following recipe.

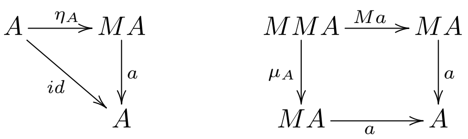
For each $\mathsf{op} \in \Sigma$, we specify a function $\alpha_{\mathsf{op}} : A^{ar(\mathsf{op})} \to A$. We call such functions the *local functions* of α. We combine all these local functions to create a function $F^{\alpha}_{\Sigma} : (A + F_{\Sigma}(A))_{\bot} \to A$ with the following definition:

$$F^{\alpha}_{\Sigma}(\bot) := \mathbf{F}_A = \bigvee\emptyset, \qquad F^{\alpha}_{\Sigma}(\mathit{inleft}(a)) = a, \qquad F^{\alpha}_{\Sigma}(\mathit{inright}(\mathsf{op}(a_1,\ldots,a_{ar(\mathsf{op})})) = \alpha_{\mathsf{op}}(a_1,\ldots,a_{ar(\mathsf{op})}).$$

This function inductively induces an algebra $\alpha : T^{\mu}_{\Sigma}(A) \to A$ on $T^{\mu}_{\Sigma}(-) = \mu Y.((-) + F_{\Sigma}(Y))_{\bot}$.

If an algebra is constructed using the above recipe, we call it a *locally constructed* algebra.

Definition 3.1 Given a monad (M, η, μ), an algebra $a : MA \to A$ is an *Eilenberg-Moore algebra* (henceforth EM-algebra) if the following two diagrams commute:



By induction on $T^{\mu}_{\Sigma}(A)$, we get the following result.

Lemma 3.2 *Any locally constructed algebra α forms an* Eilenberg-Moore algebra*.*

We use the algebra to interpret the behaviour of effectful computations, by lifting predicates on values to predicates on computations.

Given an algebra $a : MA \to A$, we can lift a quantitative predicate $P : X \to A$ to a quantitative predicate $a(P) := (a \circ M(P)) : MX \to A$. E.g., given a predicate on values $P : \mathit{Values} \to A$ we can use the aforementioned operational semantics to lift it to a predicate on computations $(\alpha(P) \circ |-|) : \mathit{Computations} \to A$.

Definition 3.3 An algebra $a : MA \to A$ on an icomplete lattice A is *leaf-ilinear* if for any set X, element $t \in T^{\mu}_{\Sigma}(X)$ and function $f : X \to \mathcal{P}_{\neq\emptyset}(A)$ associating to each $x \in X$ a non-empty subset of A,

$$\alpha(\bigvee \circ f)(t) = \bigvee\{\alpha(P)(t) \mid P : X \to A, \forall x \in X. P(x) \in f(x)\} \enspace .$$

The concept of leaf-ilinearity implements the notion of ilinearity of algebras $\alpha : T^{\mu}_{\Sigma}(A) \to A$ to the extend that it is possible, given that $T^{\mu}_{\Sigma}(A)$ is not necessarily an icomplete lattice. The leaves of $T^{\mu}_{\Sigma}(A)$ are taken from the icomplete lattice A, and as such are closed under non-empty suprema. Leaf-ilinearity asserts that the algebra preserves such suprema within the leaves.

By induction on the structure of $T^{\mu}_{\Sigma}(A)$, we have the following result.

Lemma 3.4 *If the algebra α is locally constructed by ilinear local functions, then α is leaf-ilinear.*

3.1 Infinitary trees

We use leaf-ilinearity to extend the algebra $\alpha : T^{\mu}_{\Sigma}(A) \to A$ to an algebra $\widehat{\alpha} : T^{\nu}_{\Sigma}(A) \to A$ capable of interpreting infinite trees. Firstly, note that any ilinear function is monotone. As such, we can establish the following result by induction:

Lemma 3.5 *Suppose a is a leaf-ilinear EM-algebra such that $\alpha(\bot) = \boldsymbol{F}_A$.*
For any two trees $t, t' \in T^{\mu}_{\Sigma}(A)$ such that $t \leq_{T^{\mu}_{\Sigma}(A)} t'$, $\alpha(t) \leq \alpha(t')$.

Remember from Lemma 3.4 that a locally constructed algebra α automatically satisfies the EM-algebra laws and $\alpha(\bot) = \mathbf{F}_A$ conditions.

Consider a tree $t \in T^{\nu}_{\Sigma}(A)$, then there is an ascending sequence of trees $t_0 \leq t_1 \leq t_2 \leq \dots$ in $T^{\mu}_{\Sigma}(A)$ such that $t = \bigvee_n t_n = \bigvee\{t_n \mid n \in \mathbb{N}\}$. We say that the sequence $t_0, t_1, \dots$ *approximates* t. We define $\widehat{\alpha}(t)$ as $\bigvee_n \alpha(t_n)$. This is well-defined, since for any two sequences $t_0, t_1, \dots$ and $t'_0, t'_1, \dots$ approximating t, then for any n, there is an m such that $t_n \leq t'_m$ and hence by Lemma 3.5, $\alpha(t_n) \leq \alpha(t'_m)$. Hence $\bigvee_n \alpha(t_n) \leq \bigvee_n \alpha(t'_n)$ and vice versa, $\bigvee_n \alpha(t_n) \geq \bigvee_n \alpha(t'_n)$.

Last but not least, we have the following result, for which we give a brief sketch of a proof.

Lemma 3.6 *Suppose α is a locally constructed leaf-ilinear algebra. Then $\widehat{\alpha} : T^{\nu}_{\Sigma}(A) \to A$ is an EM-algebra.*

Proof. Any ilinear function $f : A^n \to A$ is ω-continuous: Given a series of ascending sequences $\{a^1_i\}_{i\in\mathbb{N}}, \{a^2_i\}_{i\in\mathbb{N}}, \dots, \{a^n_i\}_{i\in\mathbb{N}}$ in A, then, by ilinearity and monotonicity: $f(\bigvee_i a^1_i, \dots, \bigvee_i a^n_i) = \bigvee\{f(a_{i_1}, \dots, a_{i_n}) \mid i_1, \dots, i_n \in \mathbb{N}\} = \bigvee_i f(a^1_i, \dots, a^n_i)$. Using this, we can by induction establish that, for any ascending sequence of trees $t_0 \leq_{T^{\mu}_{\Sigma}(A)} t_1 \leq_{T^{\mu}_{\Sigma}(A)} \dots$, $\widehat{\alpha}(\bigvee_{i\in\mathbb{N}} t_i) = \bigvee_{i\in\mathbb{N}} \alpha(t_i)$.

Since any tree $\eta(a)$ is finite, the η rule is simple to prove. For the μ-rule, take $d \in T^{\nu}_{\Sigma}(T^{\nu}_{\Sigma}(A))$, and take some sequence of elements $d_0, d_1, \dots$ of $T^{\mu}_{\Sigma}(T^{\mu}_{\Sigma}(A))$ approximating d. Then $\{T^{\nu}_{\Sigma}(\widehat{\alpha})(d_i)\}_{i\in\mathbb{N}}$ is an increasing sequence of elements of $T^{\nu}_{\Sigma}(A)$. Hence: $\widehat{\alpha}(T^{\nu}_{\Sigma}(\widehat{\alpha})(d)) = \widehat{\alpha}(T^{\nu}_{\Sigma}(\widehat{\alpha})(\bigvee_i d_i)) = \widehat{\alpha}(\bigvee_i T^{\mu}_{\Sigma}(\widehat{\alpha})(d_i)) = \bigvee_i \alpha(T^{\mu}_{\Sigma}(\widehat{\alpha})(d_i)) = \bigvee_i \alpha(\mu d_i) = \widehat{\alpha}(\bigvee_i \mu d_i) = \widehat{\alpha}(\mu(\bigvee_i d_i)) = \widehat{\alpha}(\mu d)$. □

In the rest of this paper, we will concern ourselves more with ilinear local functions, then with algebras. But we will keep in mind that algebras can be constructed using ilinear local functions. We specify the following structure for interpreting effects.

Definition 3.7 An *effect description* (Σ, A, α) consists of an effect signature Σ, a completely distributive lattice A, and an *interpretation* α given by an ilinear function $\alpha_{\mathsf{op}} : A^{ar(\mathsf{op})} \to A$ for each $\mathsf{op} \in \Sigma$.

3.2 Examples

We look at some examples of effects and their effect descriptions.

Example 3.8 [Probability] We consider the effect signature $\Sigma_{prob} = \{\mathsf{por}\}$ with a single effect operation por for binary probabilistic choice. The operation has arity 2, and chooses fairly between two continuations. We give this the effect description $(\Sigma_{prob}, [0,1], \mathsf{Exp})$, with as complete lattice of truth values the real number interval $[0,1]$. We give probabilistic computations the *expectation* interpretation Exp, given by the local function $\mathsf{Exp}_{\mathsf{por}} : [0,1]^2 \to [0,1]$ which calculates the average between its two arguments $\mathsf{Exp}_{\mathsf{por}}(a,b) = (a+b)/2$. The constructed algebra $\mathsf{Exp} : T^{\nu}_{\Sigma}([0,1]) \to [0,1]$ will calculate the expected value of a leaf $a \in [0,1]$, assuming each choice in a tree $t \in T^{\nu}_{\Sigma}([0,1])$ is resolved fairly.

Example 3.9 [Global Store] We consider a set of global store locations Loc for storing Boolean values. We consider the effect signature $\Sigma_{global} := \{\mathsf{lookup}_l, \mathsf{update}_l(\mathbf{T}), \mathsf{update}_l(\mathbf{F}) \mid l \in \mathsf{Loc}\}$ which for each global store location l has a lookup operation lookup_l of arity 2 and two update operations $\mathsf{update}_l(\mathbf{T})$, $\mathsf{update}_l(\mathbf{F})$ of arity 1. The update operations store a Boolean value to a global store, whereas the lookup operations look up a Boolean value from a global store location, and uses it to choose one of two continuations: The left continuation if the value is $\mathbf{T}$ and the right continuation if the value is $\mathbf{F}$. We write $\mathsf{S} := \mathbb{B}^{\mathsf{Loc}}$ for the set of *global states*, and we call elements of $\mathcal{P}(\mathsf{S}) \simeq (\mathsf{S} \to \mathbb{B})$ *assertions* on the global state. We give this effect the description $(\Sigma_{global}, \mathcal{P}(\mathsf{S}), \mathsf{Wp})$ with the *weakest precondition* interpretation Wp, where:

$\mathsf{Wp}_{\mathsf{lookup}_l}(a,b) := \{s \in a \mid s(l) = \mathbf{T}\} \cup \{s \in b \mid s(l) = \mathbf{F}\}$.

$\mathsf{Wp}_{\mathsf{update}_l(v)}(a) := \{s[l := v] \mid s \in a\}$, where $s[l := v](l) = v$ and $s[l := v](l') = s(l')$ for any $l' \neq l$.

$\mathsf{Wp} : T^{\nu}_{\Sigma}(\mathcal{P}(\mathsf{S})) \to \mathcal{P}(\mathsf{S})$ will calculate the set of correct starting states for which, when the computation is evaluated with that state, it terminates in some leaf $\langle R \rangle$ with a final state satisfying assertion $R \in \mathcal{P}(\mathsf{S})$.

Example 3.10 [Cost] We consider the effect signature $\Sigma_{cost} := \{\mathsf{cost}_q \mid q \in \mathbb{Q}_{>0}\}$, where for each positive rational number q we have an effect operation cost_q with arity 1, which requires a cost q to be spend before continuing evaluation. For example, a sleep operation which delays computation for some time, or a save operation which requires a certain amount of memory to be reserved. We see the nonnegative reals $[0,\infty]$ with reverse order as the space of *total costs* (which contain all limits of rational costs). We give the effect the description $(\Sigma_{cost}, [0,\infty], \mathsf{Tal})$ with the *tally* interpretation Tal summing all costs together, where

$\mathsf{Tal}_{\mathsf{cost}_q}(a) = a + q$.

Example 3.11 [Nondeterminism] We consider the effect signature $\Sigma_{non} = \{\mathsf{nor}\}$ with one effect operation nor for binary nondeterministic choice with arity 2. We give this the effect description $(\Sigma_{non}, \mathbb{A}, \mathsf{Pos})$, with as truth space the three element chain $\mathbb{A} := \{\mathbf{F}, \Diamond, \mathbf{T}\}$ containing the three degrees of possibility (this is e.g. used in [14] for describing nondeterminism). The smallest element $\mathbf{F}$ represents impossibility, the middle element $\Diamond$ represents possibility, and the largest element $\mathbf{T}$ represents inevitability. We give the effect the interpretation Pos which issues the degree of possibility and follows the algebraic structure established in [5,4]:

$\mathsf{Pos}_{\mathsf{nor}}(a,a) = a, \qquad \mathsf{Pos}_{\mathsf{nor}}(a,b) = \Diamond \quad \text{if } a \neq b.$

The constructed algebra $\mathsf{Pos} : T^{\nu}_{\Sigma}(\mathbb{A}) \to \mathbb{A}$ will produce $\mathbf{T}$ if any resolution of choice leads to a leaf labelled $\mathbf{T}$, otherwise it will produce $\Diamond$ if it is possible to get to a leaf labelled $\mathbf{T}$ or $\Diamond$.

Example 3.12 [Error] We consider the effect signature $\Sigma_{error} := \{\mathsf{raise}\}$ with a single effect operation raise of arity 0, which aborts evaluation displaying an error message. We use the effect description $(\Sigma_{error}, \mathbb{A}, \mathsf{Err})$ where $\mathbb{A}$ is as given above. We give this the interpretation Err where $\mathsf{Err}_{\mathsf{raise}_e}() = \Diamond$. This effect may be combined with itself using the forthcoming method for combining effects, in order to get an interpretation of multiple errors. For simplicity, we consider only one error at this time.

Example 3.13 [Pure computation] Last but not least, we consider the situation in which there is no effectful behaviour at all (except for divergence). This has the effect description $(\emptyset, \mathbb{B}, \downarrow)$ with an empty signature, the Booleans as truths space, and the *termination* interpretation, which has no local functions since the signature is empty. The constructed algebra $\downarrow : T^{\nu}_{\Sigma}(\mathbb{B}) \to \mathbb{B}$, where $T^{\nu}_{\Sigma}(\mathbb{B}) = (\mathbb{B})_{\perp}$, sends $\perp$ to $\mathbf{F}$ and $v \in \mathbb{B}$ to v.

We end this section with a short discussion on how these algebras can give rise to notions of program equivalence.

As noted before, an algebra $\alpha : T^{\nu}_{\Sigma}(A) \to A$ can be used to lift a predicate on values to a predicate on computations. This can be used to generate a logic of quantitative formulas, as done in [31]: For each type in the language, we have a collection of formulas. A program of a type functions as a model of such formulas, satisfying each formula to a certain degree $a \in A$. We say that two programs of the same type are *behaviourally equivalent* if they satisfy each formula to the same degree.

It is also possible to define a notion of *applicative bisimilarity*, in the sense of [1,7]. We can define a *relator* using our algebra, which lifts a relation on values to a relation on computations. This relator specifies a notion of *applicative bisimulation*, a relation which is closed under certain operations (like application). Two programs are *applicatively bisimular* if there is an applicative bisimulation which relates the two. Given that our algebra is an ω-continues EM-algebra, this notion of applicative bisimilarity coincides with the notion of behavioural equivalence (see [31]).

Last but not least, we can define the notion of *contextual equivalence*. We specify a *basic relation* on computations of base type: Two programs of base type are equivalent if they satisfy each predicate, lifted by the algebra, to the same degree. The contextual equivalence is then the largest *compatible* (or *congruent*) relation which, on base types, coincide with the basic relation. Contextual equivalence is a bit of an outlier since in general, A need not be a complete lattice for the definition to work, and this notion of program equivalence does not always coincide with the above two notions (in particular in the presence of nondeterminism).

4 Tensor of Complete Lattices

We have defined effect descriptions to interpret the behaviour of algebraic effects. We will now start building the foundation for combining effects and their descriptions. In particular, given two effect descriptions $(\Sigma_1, A_1, \alpha_1)$ and $(\Sigma_2, A_2, \alpha_2)$, we want to find an effect description $(\Sigma_{12}, A_{12}, \alpha_{12})$ for the combination of effects.

Firstly, the combined signature Σ_{12} is given by the sum or disjoint union of the original two signatures $\Sigma_{12} := \Sigma_1 + \Sigma_2$. Combining truth spaces and local functions is more involved. In this section, we developed the theory for combining these things, starting with the tensor operation on complete lattices. This tensor, and its two representations, are featured in [10,17,30,34].

Definition 4.1 The *tensor product* of two complete lattices A and B is a complete lattice $A \otimes B$ such that there is a universal linear function $u_{A,B} : A \times B \to A \otimes B$ with the property that: Any linear function $f : A \times B \to C$ into a complete lattice C is the composition of $u_{A,B}$ with some linear function $f^{\otimes} : (A \otimes B) \to C$.

In particular, this factorisation gives us a natural bijection between linear functions $f : (A) \times (B) \to C$ with two arguments and linear functions $g : (A \otimes B) \to C$ with one argument. This can be generalised to a bijection between linear functions $f : \Pi_{i\in I}(A_i \times B_i) \to C$ and linear functions $g : \Pi_{i\in I}(A_i \otimes B_i) \to C$.

We look at different representations of the tensor $A \otimes B$ of two complete lattices (featured in the aforementioned literature), respectively the *powerset representation* and the *function representation*:

(i) $(A \otimes B)^{\mathcal{P}} := \{S \subseteq A \times B \mid \forall x \subseteq A, y \subseteq B,\ x \times y \subseteq S \Leftrightarrow (\bigvee x, \bigvee y) \in S\}$, with inclusion order.
For any $a \in A$, $b \in B$, $u^{\mathcal{P}}_{A,B}(a, b) := \{(a', b') \in A \times B \mid a' = \mathbf{F}_A \vee b' = \mathbf{F}_B \vee (a' \leq a \ \wedge\ b' \leq b)\}$.
A linear function $f : A\times B \to C$ factors through $g : (A\otimes B)^{\mathcal{P}} \to C$ given by $g(S) := \bigvee\{f(a, b) \mid (a, b) \in S\}$.

(ii) $(A \otimes B)^{\rightarrow} := \{f : A \to B \mid \forall x \subseteq A. f(\bigvee x) = \bigwedge\{f(a) \mid a \in x\}\}$, with pointwise (extensional) order.
A linear function $f : A\times B \to C$ factors through $g : (A\otimes B)^{\rightarrow} \to C$ given by $g(h) := \bigvee\{f(a, h(a)) \mid a \in A\}$.

Depending on which two complete lattices we combine, we may choose an appropriate representation of the tensor. In the general theory, we will mainly stick to the powerset representation. Since $(A \otimes B)^{\mathcal{P}}$ and $(A \otimes B)^{\rightarrow}$ both represent the same complete lattice, there is an isomorphism between the two, given by:

- $R : (A \otimes B)^{\mathcal{P}} \to (A \otimes B)^{\rightarrow}$, $R(S) = \lambda a. \bigvee\{b \in B \mid (a, b) \in S\}$.
- $L : (A \otimes B)^{\rightarrow} \to (A \otimes B)^{\mathcal{P}}$, $L(h) = \{(a, b) \in A \times B \mid b \leq h(a)\}$.

We give a concrete definition to the aforementioned bijection between linear functions $f : \Pi_{i\in I}(A_i\times B_i) \to C$ and linear functions $g : \Pi_{i\in I}(A_i \otimes B_i) \to C$ with respect to the powerset representation of the tensor product.

$$F : (\Pi_{i\in I}(A_i \times B_i) \to C) \to (\Pi_{i\in I}(A_i \otimes B_i)^{\mathcal{P}} \to C), \quad F(f)(\lambda i.S_i) := \bigvee\{f(\lambda i.(a_i, b_i)) \mid \forall i \in I.(a_i, b_i) \in S_i\}. \tag{1}$$

We look at some known properties of the tensor product, using the two representations interchangeably.

Lemma 4.2 *The tensor product $\mathbb{B} \otimes A$, of the Booleans and a complete lattice A, is isomorphic to A.*

Proof. We use the function representation. Elements of $(\mathbb{B} \otimes A)^{\rightarrow}$ are given by supremum reversing functions $f : \mathbb{B} \to A$. These are precisely the functions $f : \mathbb{B} \to A$ such that $f(\mathbf{F}) = \mathbf{T}_A$, hence they are in one-to-one correspondence with elements of A (values given by $f(\mathbf{T})$). ☐

Proposition 4.3 *The tensor product gives a symmetric monoidal product in the category of complete lattices Com, with the unit given by the Boolean $\mathbb{B}$.*

Proof. First note that the powerset representation immediately gives us symmetry. Using the function representation, it can be shown that $(A\otimes(B\otimes C)^{\rightarrow})^{\rightarrow}$ is isomorphic to $(B\otimes(A\otimes C)^{\rightarrow})^{\rightarrow}$. Using these isomorphisms: $(A \otimes (B \otimes C)) \simeq (A \otimes (C \otimes B)) \simeq (A \otimes (C \otimes B)^{\rightarrow})^{\rightarrow} \simeq (C \otimes (A \otimes B)^{\rightarrow})^{\rightarrow} \simeq (C \otimes (A \otimes B)) \simeq ((A \otimes B) \otimes C)$.
Lemma 4.2 shows the Booleans are a unit for the tensor product. ☐

We look at one more example of a tensor product relevant to combining effects (Example 1.2.9 from [9]).

Lemma 4.4 *The tensor product $(\mathcal{P}(S)\otimes A)$, of the powerset lattice $\mathcal{P}(S)$ and a complete lattice A, is isomorphic to the complete lattice $(S \to A)$, of functions from S to A, with pointwise order.*

Proof. We use the function representation. Take $f \in (\mathcal{P}(S) \otimes A)^{\to}$, which is a supremum reversing function $f : \mathcal{P}(S) \to A$. Then for any set $K \subseteq S$, $f(K) = f(\bigcup_{s\in K}\{s\}) = \bigwedge_{s\in K} f(\{s\})$. Hence f is completely determined by the function $f' : S \to A$ given by $\lambda s.f(\{s\})$. Vice versa, each function $g : S \to A$ determines an f sending K to $\bigwedge_{s\in K} g(s)$. Hence $(\mathcal{P}(S) \otimes A)^{\to}$ is isomorphic to the function space $S \to A$. □

4.1 The supremum of tensor products

We have a closer look at the powerset representation of the tensor of two complete lattices. It is easy to establish that infimum operation on $(A \otimes B)^{\mathcal{P}}$ is given by the intersection on sets. The supremum operation on $(A \otimes B)^{\mathcal{P}}$ however is more complicated. Luckily, if the tensor is taken over completely distributive lattices, this supremum can be given a clear and usable characterisation. This will be useful later in the paper.

Firstly, we investigate the following closure operation.

Definition 4.5 Given two complete lattices A and B, and a subset $S \subseteq A \times B$, let $\widehat{S} \subseteq A \times B$ be the set $\{(\bigvee x, \bigvee y) \mid x \subseteq A, y \subseteq B, x \times y \subseteq S\}$.

Note that if $S \in (A \otimes B)$, then $S = \widehat{S}$. Moreover, for $S \subseteq S' \subseteq A \times B$, $\widehat{S} \subseteq \widehat{S'}$, hence $\widehat{S}$ is always included in the smallest element of $(A \otimes B)$ containing S. With the following lemma, we can prove that under certain conditions, $\widehat{S}$ is the smallest element of $(A \otimes B)$ including S.

We call a subset $S \subseteq A \times B$ *down-closed* if for any $(a, b) \in S$, $a' \leq a$, and $b' \leq b$, we get $(a', b') \in S$.

Lemma 4.6 *Suppose that both A and B are completely distributive lattices, then for any down-closed $S \subseteq A \times B$, $\widehat{S} \in (A \otimes B)$.*

Proof. Suppose $(\bigvee x, \bigvee y) \in \widehat{S}$, then there are $x' \subseteq A$ and $y' \subseteq B$ such that $(x' \times y') \subseteq S$, $\bigvee x' = \bigvee x$ and $\bigvee y' = \bigvee y$. Hence for any $a \in x$ and $b \in y$, by down-closure of S, $(\{a\wedge a' \mid a' \in x'\}\times\{b\wedge b' \mid b' \in y'\}) \subseteq S$, hence $(\bigvee\{a \wedge a' \mid a' \in x'\}, \bigvee\{b\wedge b' \mid b' \in y'\}) \in \widehat{S}$. Now, by distributivity, $\bigvee\{a \wedge a' \mid a' \in x'\} = a \wedge \bigvee x' = a \wedge \bigvee x = a$ and similarly $\bigvee\{b \wedge b' \mid b' \in y'\} = b$. Hence $(a, b) \in \widehat{S}$.

Suppose $(x \times y) \subseteq \widehat{S}$, then for any $a \in x$ and $b \in y$, there are $x^b_a \subseteq A$ and $y^b_a \subseteq B$ such that $(x^b_a \times y^b_a) \subseteq S$, $\bigvee x^b_a = a$, and $\bigvee y^b_a = b$. For a family of sets of elements $\{z_i\}_{i\in I}$ of a complete lattice, we denote $\bigwedge_i z_i$ for the set $\{\bigwedge_{i\in I} c_i \mid \forall i \in I, c_i \in z_i\}$. Since S is down-closed, it holds that for any $a \in x$, $b' \in y$, $((\bigwedge_{b\in y} x^b_a) \times y^{b'}_a) \subseteq S$, hence for any $a \in x$, $((\bigwedge_{b\in y} x^b_a) \times (\bigcup_{b\in y} y^b_a)) \subseteq S$ (where $\bigcup$ is union). Repeating this process, we can derive that $((\bigcup_{a\in x}(\bigvee_{b\in y} x^b_a)) \times (\bigwedge_{a\in x}(\bigcup_{b\in y} y^b_a))) \subseteq S$, hence $(\bigvee(\bigcup_{a\in x}(\bigwedge_{b\in y} x^b_a)), \bigvee(\bigwedge_{a\in x}(\bigcup_{b\in y} y^b_a))) \in \widehat{S}$. Now, by distributivity,

$$\bigvee(\textstyle\bigcup_{a\in x}(\bigwedge_{b\in y} x^b_a)) = \bigvee_{a\in x}(\bigvee(\bigwedge_{b\in y} x^b_a)) = \bigvee_{a\in x}(\bigwedge_{b\in y}(\bigvee x^b_a)) = \bigvee_{a\in x}(\bigwedge_{b\in y} a) = \bigvee_{a\in x} a = \bigvee x,$$
$$\bigvee(\textstyle\bigwedge_{a\in x}(\bigcup_{b\in y} y^b_a)) = \bigwedge_{a\in x}(\bigvee(\bigcup_{b\in y} y^b_a)) = \bigwedge_{a\in x}(\bigvee_{b\in y}(\bigvee y^b_a)) = \bigwedge_{a\in x}(\bigvee_{b\in y} b) = \bigwedge_{a\in x}(\bigvee y) = \bigvee y.$$

We conclude that $(\bigvee x, \bigvee y) \in \widehat{S}$. □

Corollary 4.7 *For A and B two completely distributive lattices, then for any $X \subseteq (A \otimes B)$, $\bigvee X = \widehat{\bigcup X}$.*

Proof. Since all members of X, as elements of $(A \otimes B)$, are down-closed, we know $\bigcup X$ is down-closed as well. Hence by the previous lemma, $\widehat{\bigcup X} \in (A \otimes B)$. So, considering $\bigvee X$ must be the smallest element of $(A \otimes B)$ containing $\bigcup X$, it must be equal to $\widehat{\bigcup X}$. □

An example: $u_{A,B}(a, b) \vee u_{A,B}(a', b') = u_{A,B}(a, b) \cup u_{A,B}(a', b') \cup u_{A,B}(a \vee a', b \wedge b') \cup u_{A,B}(a \wedge a', b \vee b')$.

5 Combining functions

The tensor product gives us a clear way of combining two linear functions $f : \Pi_{i\in I}A \to A$ and $g : \Pi_{i\in I}B \to B$ into a single linear function $(f \otimes g) : \Pi_{i\in I}(A \otimes B) \to (A \otimes B)$. This is done in the following way:

(i) Compose f and g with $u_{A,B}$ into a single linear function $u_{A,B} \circ (f, g) : \Pi_{i\in I}A \times \Pi_{i\in I}B \to (A \otimes B)$.

(ii) Permute the arguments of the function to get a linear function from $\Pi_{i\in I}(A \times B)$ to $(A \otimes B)$.

(iii) Apply F from 1 to the result to get a linear function $(f \otimes g) : \Pi_{i\in I}(A \otimes B) \to (A \otimes B)$.

To get a clearer picture of what is going on, we can apply this construction directly to the powerset representation $(A \otimes B)^{\mathcal{P}}$ of the tensor product. This gives us a concrete definition of the combination of functions $(f \otimes g)^{\mathcal{P}} : \Pi_{i\in I}(A \otimes B)^{\mathcal{P}} \to (A \otimes B)^{\mathcal{P}}$, which is as follows:

$$(f \otimes g)^{\mathcal{P}}(\lambda i.S_i) \quad := \quad \bigvee\{u_{A,B}(f(\lambda i.a_i), g(\lambda i.b_i)) \mid \forall i.(a_i, b_i) \in S_i\}.$$

Using the powerset representation, we can make the following observation.

Proposition 5.1 *If both A and B are completely distributive lattices, then $A \otimes B$ is completely distributive.*

Proof. We use the alternative definition for complete distributivity given in Definition 2.4.

Take some set I, and consider the infimum functions $\bigwedge_I^A$ and $\bigwedge_I^B$ on A and B respectively. Since A and B are completely distributive, both infimum functions are linear. Hence their combination $(\bigwedge_I^A \otimes \bigwedge_I^B)$ is linear too. Consider some family $\{S_i\}_{i\in I}$ of elements of $(A \otimes B)$.

$$\begin{aligned}
(\textstyle\bigwedge_I^A \otimes \bigwedge_I^B)^{\mathcal{P}}(\lambda i.S_i) &= \textstyle\bigvee\{u_{A,B} \circ (\bigwedge_I^A, \bigwedge_I^B)(\lambda i.a_i, \lambda i.b_i) \mid \forall i.(a_i, b_i) \in S_i\} \\
&= \textstyle\bigvee\{u_{A,B}(\bigwedge_{i\in I} a_i, \bigwedge_{i\in I} b_i) \mid \forall i.(a_i, b_i) \in S_i\} \\
&= \textstyle\bigvee\{u_{A,B}(a, b) \mid (a, b) \in \bigcap_{i\in I} S_i\} \\
&= \textstyle\bigcap_{i\in I} S_i = \bigwedge_{i\in I} S_i
\end{aligned}$$

This used the facts that, for $X \subseteq (A \otimes B)$, $\bigcap X = \bigwedge X$, and for $S \in (A \otimes B)$, $\bigvee\{u_{A,B}(a, b) \mid (a, b) \in S\} = S$.

Hence, $(\bigwedge_I^A \otimes \bigwedge_I^B)$ is the infimum function on $(A \otimes B)$, which we know is linear. This holds for all I, so we conclude by Definition 2.4 that $(A \otimes B)$ is completely distributive. □

5.1 Combining ilinear functions

As can be seen in the examples of Subsection 3.2, local functions are not linear in general, they are ilinear. For example, $\mathsf{Exp}_{\mathsf{por}}(\mathbf{F}_{[0,1]}, \mathbf{T}_{[0,1]}) = (0 + 1)/2 \neq 0 = \mathbf{F}_{[0,1]}$. So we need a method of combining ilinear functions. Luckily, if all the complete lattices concerned are completely distributive, the bijection F given in (1) preserves ilinearity. This allows us to use the same construction to combine ilinear functions.

Lemma 5.2 *Suppose that A and B are completely distributive lattices, then for any ilinear function $f : \Pi_{i\in I}(A \times B) \to C$, $F(f)$, the function $F(f) : \Pi_{i\in I}(A \otimes B) \to C$ is ilinear.*

Proof. Take for any $i \in I$ an arbitrary set $X_i \subseteq (A \otimes B)$. By Corollary 4.7, $\forall i \in I. \bigvee X_i = \widehat{\bigcup X_i}$. Since for any $a \in A$ and $b \in B$, $\bigcup X_i$ contains $(a, \mathbf{F}_B)$ and $(\mathbf{F}_A, b)$, it already contains $(\bigvee x, \bigvee y)$ for either x or y empty, hence we have $\bigvee X_i = \widehat{\bigcup X_i} = \{(\bigvee x, \bigvee y) \mid x \subseteq A, y \subseteq B, \text{both non-empty and } x \times y \subseteq \bigcup X_i\}$.

$$\begin{aligned}
F(f)(\lambda i. \textstyle\bigvee X_i) &= \textstyle\bigvee\{f(\lambda i.(a_i, b_i)) \mid \forall i \in I.(a_i, b_i) \in \bigvee X_i\} \\
&= \textstyle\bigvee\{f(\lambda i.(\bigvee x_i, \bigvee y_i)) \mid \forall i \in I.x_i \subseteq A, y_i \subseteq B, \text{both non-empty s.t. } x_i \times y_i \subseteq \bigcup X_i\} \\
(\textit{by ilinearity}) \quad &= \textstyle\bigvee\{f(\lambda i.(a_i, b_i)) \mid \forall i \in I.a_i \in x_i \subseteq A, b_i \in y_i \subseteq B, x_i \times y_i \subseteq \bigcup X_i\} \\
&= \textstyle\bigvee\{f(\lambda i.(a_i, b_i)) \mid \forall i \in I.(a_i, b_i) \in \bigcup X_i\} \\
&= \textstyle\bigvee\{f(\lambda i.(a_i, b_i)) \mid \forall i \in I.\exists S_i \in X_i.(a_i, b_i) \in S_i\} \\
&= \textstyle\bigvee\{\bigvee\{f(\lambda i.(a_i, b_i)) \mid \forall i.(a_i, b_i) \in S_i\} \mid \forall i.S_i \in X_i\} \\
&= \textstyle\bigvee\{F(f)(\lambda i.S_i) \mid \forall i \in I.S_i \in X_i\} \ .
\end{aligned}$$

We conclude that $F(f) : \Pi_{i\in I}(A \otimes B) \to C$ is ilinear. □

We finish this section with this final important result, showing that we can safely combine ilinear functions.

Proposition 5.3 *Suppose A and B are two completely distributive lattices, and consider two ilinear functions $f : (\Pi_{i\in I}A) \to A$ and $g : (\Pi_{i\in I}B) \to B$. Then $(f \otimes g) : (\Pi_{i\in I}(A \otimes B)) \to (A \otimes B)$ given by $F(u_{A,B} \circ (f, g))$ is ilinear.*

Proof. Note that $u_{A,B}$ is linear, hence ilinear. So $(\Pi_{i\in I}u_{A,B}) \circ (f, g)$ is ilinear as well. Hence we get the result by a simple application of Lemma 5.2. □

There is an alternative, but equivalent, formulation for the above combination of ilinear functions. We define the tensor $\otimes^i$ of icomplete lattices, the *itensor*, using Definition 4.1 but replacing linearity with ilinearity. This gives us a bijective function $F^i : \Pi_{i\in I}(A \times B) \to (A \otimes^i B)$ between spaces of ilinear function. Using the adjunction $U \vdash !$, we can find an ilinear function $(UA \otimes^i UB) \to U(A \otimes B)$ over complete lattices A and B. We compose F^i with that function to retrieve our function F. Proving that these functions coincide however, seems to require quite a bit of effort. As such, the approach in this paper uses the more direct, and theoretically less cumbersome, approach. Further comparison between the two tensors could be an interesting research topic for the future.

6 Combining effect descriptions

We combine two effect descriptions $(\Sigma_1, A_1, \alpha_1)$ and $(\Sigma_2, A_2, \alpha_2)$ into an effect description $(\Sigma_{12}, A_{12}, \alpha_{12})$, where $\Sigma_{12} := \Sigma_1 + \Sigma_2$, $A_{12} := (A_1 \otimes A_2)$, and α_{12} is defined by the following ilinear local functions:

$$\text{For } \mathsf{op} \in \Sigma_1, \text{ let } \alpha_{12,inleft(\mathsf{op})} : (A \otimes B)^{ar(\mathsf{op})} \to (A \otimes B) \text{ be } (\alpha_{1,\mathsf{op}} \otimes \textstyle\bigwedge_{ar(\mathsf{op})}).$$
$$\text{For } \mathsf{op} \in \Sigma_2, \text{ let } \alpha_{12,inright(\mathsf{op})} : (A \otimes B)^{ar(\mathsf{op})} \to (A \otimes B) \text{ be } (\textstyle\bigwedge_{ar(\mathsf{op})} \otimes \alpha_{2,\mathsf{op}}).$$

The combined effect interpretation is defined by combining the relevant interpretation of an effect operation from its source with the infimum function. Hence, since A_1 and B_1 are completely distributive, α_{12} is made up of ilinear local functions. For ease of notation, we will write $\alpha_1 * \alpha_2$ for α_{12} when looking at examples.

Lemma 6.1 *Given some ilinear function $f : A^I \to A$ and infimum/conjunction $\bigwedge : \mathbb{B}^I \to \mathbb{B}$ on the Booleans. Then $(f \otimes \bigwedge)$ is under the isomorphism $(A \otimes \mathbb{B}) \simeq A$ from Lemma 4.2 equal to f.*

Proof. $(f \otimes \bigwedge)(\lambda i.S_i) = \bigvee\{u_{A,\mathbb{B}}(f(\lambda i.a_i), \bigwedge_{i \in I} b_i) \mid \forall i \in I.(a_i, b_i) \in S_i\}$. Since for any $S \in (A \otimes \mathbb{B})$ and all $a \in S$, $(a, \mathbf{F}) \in S$, we need only concern ourselves with the case where $\bigwedge_{i \in I} b_i \neq \mathbf{F}$, hence when $\forall i \in I.b_i = \mathbf{T}$. So $(f \otimes \bigwedge)(\lambda i.S_i) = \bigvee\{u_{A,\mathbb{B}}(f(\lambda i.a_i), \mathbf{T}) \mid \forall i \in I.(a_i, \mathbf{T}) \in S_i\}$. Using the isomorphism $(A \otimes \mathbb{B}) \simeq A$ we can transport $(f \otimes \bigwedge)$ to a function from A^I to A, which is given by sending $\lambda i.a_i$ to:
$\bigvee\{a \mid (a, \mathbf{T}) \in \bigvee\{u_{A,\mathbb{B}}(f(\lambda i.a'_i), \mathbf{T}) \mid \forall i \in I.(a'_i, \mathbf{T}) \in u_{A,\mathbb{B}}(a_i, \mathbf{T})\}\} = f(\lambda i.a_i)$. ◻

In general, for $\mathsf{op} \in \Sigma_1$, $\alpha_{12,\mathsf{op}}(\lambda i.S_i) = \bigvee\{u_{A,B}(\alpha_{1,\mathsf{op}}(\lambda i.a_i), b) \mid b \in B, \forall i.(a_i, b) \in S_i\}$.

Proposition 6.2 *The method of combining effects gives a symmetric and associative operation on effect descriptions, with a unit given by pure computations $(\emptyset, \mathbb{B}, \downarrow)$.*

Proof. The sum on signatures Σ is a symmetric and associative operation, with a unit given by the empty set $\emptyset$. By Proposition 4.3, we know the tensor to be a symmetric and associative operation on complete lattices, with a unit given by $\mathbb{B}$. Lastly, the tensor combination of ilinear functions is symmetric, associative [3], and has a unit given by an infimum function over $\mathbb{B}$ (see Lemma 6.1). ◻

6.1 Examples of combining effects

To illustrate how the above method yields valid interpretations of combinations of effects, we look at a handful of examples. In each case, we add a specific effect to an arbitrary effect description. In Subsection 7.1, we will look at some more specific combinations of effects.

Example 6.3 [Adding nondeterminism] Take some effect description (Σ, A, α). To this, we add nondeterminism with choice operator $\mathsf{nor} : \alpha^2 \to \alpha$, truth space $\mathbb{A}$ and interpretation Pos. The combined truth space $(A \otimes \mathbb{A})$ is given by the space of pairs $\{(a, b) \in A^2 \mid a \leq b\}$. Such a pair (a, b) represents a worst-case value a and a best case value b, corresponding to demonic and angelic nondeterminism respectively.

Let β be the combined interpretation given by $\alpha * \mathsf{Pos}$. Our method yields the following local functions:

- $\forall \mathsf{op} \in \Sigma.\ \beta_{inleft(\mathsf{op})}(\lambda i.(a.i, b.i)) \ = \ (\alpha_{\mathsf{op}}(\lambda i.a_i), \alpha_{\mathsf{op}}(\lambda i.b_i))$.
- $\beta_{inright(\mathsf{nor})}((a_1, b_1), (a_2, b_2)) \ = \ (a_1 \wedge a_2, b_1 \vee b_2)$.

We can give β an alternative description. The algebra β will resolve nondeterministic choices of an input tree both in the worst possible way and in the best possible way. It will then apply α to the resulting two trees, and return two values of A respectively.

Example 6.4 [Adding Global Store] Take some effect description (Σ, A, α). To this we add Boolean Global Store with effect signature $\Sigma_{global} = \{\mathsf{lookup}_l, \mathsf{update}_l(\mathbf{T}), \mathsf{update}_l(\mathbf{F}) \mid l \in \mathsf{Loc}\}$, assertions $\mathcal{P}(\mathsf{S})$ and weakest precondition interpretation Wp. As shown in Lemma 4.4, the tensor of the two complete lattices is given by the function space $\mathsf{S} \to A$. We can see this as the space of *valuations* or quantitative A-valued assertions on global states S. Let β be the combination of interpretations $(\alpha * \mathsf{Wp})$, then we get:

[3] This fact is not completely trivial, but is straightforward to prove.

- $\forall \mathsf{op} \in \Sigma.\ \beta_{inleft(\mathsf{op})}(\lambda i.f_i) = \lambda s \in \mathsf{S}.\alpha_{\mathsf{op}}(\lambda i.f_i(s))$.
- $\beta_{inright(\mathsf{lookup}_l)}(f_1, f_2) = \lambda s \in \mathsf{S}.(f_1(s) \quad \text{if} \quad s(l) = \mathbf{T}, \quad \text{else} \quad f_2(s))$.
- $\beta_{inright(\mathsf{update}_l(v))}(f) = \lambda s \in \mathsf{S}.f(s[l := v])$.

We can give β an alternative description. The algebra β will resolve, for each starting state s, the global store operations of its input tree appropriately, and give its leaves (which are functions from S to A) the final state as argument. To each resulting tree, it will apply α to reach the appropriate value in A.

Example 6.5 [Adding Cost] Take some effect description (Σ, A, α). To this, we add the cost effect, with effect signature $\Sigma_{cost} := \{\mathsf{cost}_q \mid q \in \mathbb{Q}_{\geq 0}\}$, truth space $[0, \infty]$ with reverse order, and the tally interpretation Tal. The tensor of the truth spaces is given by functions $\mathbb{R}_{\geq 0} \to A$ which are infimum preserving with respect to the standard ordering of the real numbers. Technically, it is functions from $[0, \infty]$, but ∞ is always sent to $\mathbf{T}_A$, so we can remove it from the definition without loss of generality.

Let $\beta = (\alpha * \mathsf{Tal})$ be the combination of interpretations. Then:

- For $\mathsf{op} \in \Sigma$, $\beta_{inleft(\mathsf{op})}(\lambda i.f_i) = \lambda r \in \mathbb{R}_{\geq 0}.\ \alpha_{\mathsf{op}}(\lambda i.f_i(r))$.
- $\beta_{inright(\mathsf{cost}_q)}(f) = \lambda r \in \mathbb{R}_{\geq 0}.\ f(r + q)$.

We can give β an alternative description. The algebra β will, given a certain allowance $r \in \mathbb{R}_{\geq 0}$ to spend, go through the tree spending the allowance on resolving any cost operations. Once it encounters a cost operation it cannot pay for, it puts a $\perp$ leaf at that location. If it encounters another leaf, which contains a function from $\mathbb{R}_{\geq 0}$ to A, it feeds this function the remaining allowance as an argument and puts the result as the new value of the leaf. It then it applies α to the resulting tree.

7 Equations and interaction laws

Given some countable set of variables $\mathbb{V}$, we see an element of $T^{\nu}_{\Sigma}(\mathbb{V})$ as an *algebraic expression*. A pair of algebraic expressions $(e_1, e_2) \in T^{\nu}_{\Sigma}(\mathbb{V}) \times T^{\nu}_{\Sigma}(\mathbb{V})$ expresses an *equation* ($e_1 = e_2$) or *inequation* ($e_1 \leq e_2$). A relation on algebraic expressions $\mathcal{R} \subseteq (T^{\nu}_{\Sigma}(\mathbb{V}))^2$ can be specified by choosing a set of axioms $\mathcal{A} \subseteq (T^{\nu}_{\Sigma}(\mathbb{V}))^2$ appropriate to the effect, and closing this set under a couple of proof rules, e.g. *compositionality*:

$$\forall (e_1, e_2) \in \mathcal{R}, \forall f, g : \mathbb{V} \to T^{\nu}_{\Sigma}(\mathbb{V}). \quad (\forall x \in \mathbb{V}. f(x) \mathcal{R} g(x)) \implies (T^{\nu}_{\Sigma}(f)(e_1), T^{\nu}_{\Sigma}(g)(e_2)) \in \mathcal{R}\ .$$

An axiom we tend to assume for each effect is the inequation ($\perp \leq x$) where $x \in \mathbb{V}$. This reflects the fact that a diverging computation does not produce anything observable.

An EM-algebra $\alpha : T^{\nu}_{\Sigma}(A) \to A$ on some preorder specifies a relation $\mathcal{R}_\alpha \subseteq (T^{\nu}_{\Sigma}(\mathbb{V}))^2$ as follows:

$$(e_1, e_2) \in \mathcal{R}_\alpha \quad \iff \quad \forall f : \mathbb{V} \to A.\ \widehat{\alpha}(T^{\nu}_{\Sigma}(f)(e_1)) \leq \widehat{\alpha}(T^{\nu}_{\Sigma}(f)(e_2))\ .$$

If α is an EM-algebra, then $\mathcal{R}_\alpha$ is compositional. More on this comparison can be found in [33].

We say that an algebra α *complements* a set of axioms $\mathcal{A}$ if they generate the same algebraic relation. The word complement expresses the fact that, whereas equations state equality between programs, algebras are used to find distinctions between programs (see Examples 7.2 and 7.6 for such a distinctions). An equation (e_1, e_2) holds for α if $(e_1, e_2) \in \mathcal{R}_\alpha$. This direct correspondence allows us to compare the method of combining effects defined in this paper with traditional methods for combining equational theories of effects [15,16].

Firstly, we observe that equations which holds for the individual effects still holds for the combination of effects. Note that for two effect signatures $\Sigma \subseteq \Sigma'$, $T^{\nu}_{\Sigma}(\mathbb{V}) \subseteq T^{\nu}_{\Sigma'}(\mathbb{V})$. For simplicity, we consider $\Sigma_1 + \Sigma_2 = \Sigma_1 \cup \Sigma_2$, hence it contains both Σ_1 and Σ_2.

Proposition 7.1 *Given two effect descriptions $(\Sigma_1, A_1, \alpha_1)$ and $(\Sigma_2, A_2, \alpha_2)$, let $(\Sigma_{12}, A_{12}, \alpha_{12})$ be their combination. Then both $\mathcal{R}_{\alpha_1} \subseteq T^{\nu}_{\Sigma_1}(\mathbb{V})$ and $\mathcal{R}_{\alpha_2} \subseteq T^{\nu}_{\Sigma_1}(\mathbb{V})$ are contained in $\mathcal{R}_{\alpha_{12}} \subseteq T^{\nu}_{\Sigma_{12}}(\mathbb{V})$.*

Proof. Consider $t \in T^{\nu}_{\Sigma_1}((A \otimes B)^{\mathcal{P}})$, hence t only has internal nodes from Σ_1. For $b \in B$, we define the function $f_b : (A \otimes B)^{\mathcal{P}} \to A$ by $S \mapsto \bigvee\{a \in A \mid (a, b) \in S\}$. We prove that $\widehat{\alpha_{12}}(t) = \bigvee\{u_{A,B}(\widehat{\alpha_1}(T^{\nu}_{\Sigma_1}(f_b)(t)), b) \mid b \in B\}$. We start with an induction over finite trees, using the local functions. First the base cases.

If $t = \perp$, then $\alpha_{12}(t) = \mathbf{F}_{A,B} = \bigvee\{u_{A,B}(\mathbf{F}_A, b) \mid b \in B\} = \bigvee\{u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t)), b) \mid b \in B\}$.

If $t = \langle S \rangle$, then $\alpha_{12}(t) = S = \bigvee\{u_{A,B}(a, b) \mid (a, b) \in S\} = \bigvee\{u_{A,B}(f_b(S), b) \mid b \in B\} = \bigvee\{u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t)), b) \mid b \in B\}$.

If $t = \mathsf{op}\langle t_1, \ldots, t_n \rangle$, then

$$\begin{aligned}
\alpha_{12}(t) = \quad & \alpha_{12,\mathsf{op}}(\alpha_{12}(t_1),\ldots,\alpha_{12}(t_n)) \\
= \quad & \alpha_{12,\mathsf{op}}(\bigvee\{u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t_1)),b) \mid b\in B\},\ldots,\bigvee\{u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t_n)),b)\mid b\in B\}) \\
= \quad & \bigvee\{\alpha_{12,\mathsf{op}}(u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_{b_1})(t_1)),b_1),\ldots,u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_{b_n})(t_n)),b_n)) \mid b_1,\ldots,b_n\in B\} \\
= \quad & \bigvee\{u_{A,B}(\alpha_{1,\mathsf{op}}(a_1,\ldots,a_n),\textstyle\bigwedge_i b'_i) \mid \forall i.b_i\in B,(a_i,b'_i)\in u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_{b_i})(t_i)),b_i)\} \\
= \quad & \bigvee\{u_{A,B}(\alpha_{1,\mathsf{op}}(a_1,\ldots,a_n),b) \mid \forall i,b_i\in B,b\in B,(a_i,b)\in u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_{b_i})(t_i)),b_i)\} \\
= \quad & \bigvee\{u_{A,B}(\alpha_{1,\mathsf{op}}(a_1,\ldots,a_n),b) \mid \forall i,a_i\in A,b\in B,a_i\le\alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t_i))\} \\
= \quad & \bigvee\{u_{A,B}(\alpha_{1,\mathsf{op}}(\alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t_1)),\ldots,\alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t_n))),b)\mid b\in B\} \\
= \quad & \bigvee\{u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t_1)),b)\mid b\in B\} \quad .
\end{aligned}$$

Which is what we wanted to prove. For an infinite tree t approximated by finite trees $t_1, t_2, t_3, \ldots$:
$\widehat{\alpha_{12}}(t) = \widehat{\alpha_{12}}(\bigvee_i t_i) = \widehat{\alpha_{12}}(\bigvee_i t_i) = \bigvee_i \alpha_{12}(t_i) = \bigvee_i \bigvee\{u_{A,B}(\alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t_i)),b) \mid b\in B\} = \bigvee\{u_{A,B}(\bigvee_i \alpha_1(T^{\mu}_{\Sigma_1}(f_b)(t_i)),b)\mid b\in B\} = \bigvee\{u_{A,B}(\widehat{\alpha_1}(T^{\nu}_{\Sigma_1}(f_b)(\bigvee_i t_i)),b)\mid b\in B\}$.

Consider an equation $(e_1,e_2)\in(T^{\nu}_{\Sigma_1}(\mathbb{V}))$ which holds for α_1. Then, given $f:\mathbb{V}\to(A\otimes B)$, we know by the above result that $\widehat{\alpha_{12}}(T^{\nu}_{\Sigma_1}(f)(e_1)) = \widehat{\alpha_{12}}(T^{\nu}_{\Sigma_1}(f)(e_2))$, hence the equation holds for α_{12}. □

In particular, if α_1 complements a set of axioms $\mathcal{A}$, then all equations from $\mathcal{A}$ still hold for α_{12}. We conclude that equations are preserved.

7.1 Comparing methods of combining effects

Given two sets of axioms $\mathcal{A}_1 \subseteq (T^{\nu}_{\Sigma_1}(\mathbb{V}))^2$ and $\mathcal{A}_2 \subseteq (T^{\nu}_{\Sigma_2}(\mathbb{V}))^2$, then we define the axioms of the *equational sum* [15,16] of effects to be $\mathcal{A}_1\cup\mathcal{A}_2$. This is called the sum, since it only contains the original axioms. In some cases, the interpretation of a combination of effects defined in this paper corresponds to the equational sum:

Example 7.2 [Cost with Error] We combine the effects of cost in Example 3.10, and error in Example 3.12, using Example 6.5. The resulting combination of effects coincides with the sum of equational theories. We can see that for any $q\in\mathbb{Q}_{>0}$, the effect operation cost_q does not distribute over raise. This is because $\mathsf{cost}_q(\mathsf{raise}())$ and $\mathsf{raise}()$ are distinguished by the combined algebra $(\mathsf{Tal}*\mathsf{Err})$:

$$(\mathsf{Tal}*\mathsf{Err})(\mathsf{raise}())(0) \quad = \quad \Diamond \quad \neq \quad \mathbf{F} \quad = \quad (\mathsf{Tal}*\mathsf{Err})(\mathsf{cost}_q(\mathsf{raise}()))(0) \quad .$$

Or informally, if we have no resources (the 0 argument), then $\mathsf{raise}()$ will yield an error (the $\Diamond$), whilst $\mathsf{cost}_q(\mathsf{raise}())$ will stall (the $\bot$) as it requests a resource we do not have. We can use this as a basis to prove that our combined algebra complements the sum of equational theories..

In other cases, extra axioms are needed to describe the interaction between the effects we want to combine. Such axioms, which contain effect operations from both theories, are called *interaction laws*. The most common interaction law is a *commutativity law* [15,16].

Definition 7.3 Given two effect operations op_1 and op_2 of arity n and m respectively, the *commutativity* law between op_1 and op_2 is given by:

$$\mathsf{op}_1(\lambda i.\ \mathsf{op}_2(\lambda j.\ v_{i,j})) \ = \ \mathsf{op}_2(\lambda j.\ \mathsf{op}_1(\lambda i.\ v_{i,j})) \ ,$$

where we use a distinct variable $v_{i,j}\in\mathbb{V}$ for each $1\le i\le n$ and $1\le j\le m$.[4]

Let $\mathsf{commut}(\Sigma_1,\Sigma_2)\subseteq T^{\nu}_{\Sigma_1+\Sigma_2}(\mathbb{V})$ be the set containing the commutativity law for each pair $\mathsf{op}_1\in\Sigma_1$ and $\mathsf{op}_2\in\Sigma_2$. Given two sets of axioms $\mathcal{A}_1\subseteq(T^{\nu}_{\Sigma_1}(\mathbb{V}))^2$ and $\mathcal{A}_2\subseteq(T^{\nu}_{\Sigma_2}(\mathbb{V}))^2$, then we define the axioms of the *equational tensor* [16] of effects to be $\mathcal{A}_1\cup\mathcal{A}_2\cup\mathsf{commut}(\Sigma_1,\Sigma_2)$. In some cases, the description of a combination of effects as defined in this paper corresponds to the equational tensor:

Example 7.4 [Probability with Global store] We combine the effects of probability in Example 3.8, and global store in Example 3.9, using Example 6.4. We will show that the relation $\mathcal{R}_\alpha$ induced by the combined algebra $\alpha=\mathsf{Exp}*\mathsf{Wp}$ contains all the commutativity laws. From Example 6.4, we know that the combined truth space is given by $A=(\mathsf{S}\to[0,1])$. Now, por commutes with lookup_l since, given $a,b,c,d\in A$:

[4] The choice of variables is unimportant, since the resulting (induced) relation on $T^{\nu}_{\Sigma}(\mathbb{V})$ will be closed under substitution.

$$\begin{aligned}\alpha(\mathsf{por}(\mathsf{lookup}_l(a,b),\mathsf{lookup}_l(c,d))) &= \lambda s.(\alpha(\mathsf{lookup}_l(a,b))(s)+\alpha(\mathsf{lookup}_l(c,d))(s))/2\\ &= \lambda s.\begin{cases}(a(s)+c(s))/2 & \text{if } s(l)=\mathbf{T}\\ (b(s)+d(s))/2 & \text{if } s(l)=\mathbf{F}\end{cases}\\ &= \alpha(\mathsf{lookup}_l(\mathsf{por}(a,c),\mathsf{por}(b,d)))\quad .\end{aligned}$$

The operation por commutes with $\mathsf{update}_{l,\mathbf{T}}$ since, given $a,b\in A$:

$$\begin{aligned}\alpha(\mathsf{por}(\mathsf{update}_{l,\mathbf{T}}(a),\mathsf{update}_{l,\mathbf{T}}(b))) &= \lambda s.(\alpha(\mathsf{update}_{l,\mathbf{T}}(a))(s)+\alpha(\mathsf{update}_{l,\mathbf{T}}(b))(s))/2\\ &= \lambda s.(a(s[l:=\mathbf{T}])+b(s[l:=\mathbf{T}]))/2\\ &= \lambda s.\alpha(\mathsf{por}(a,b))(s[l:=\mathbf{T}])\\ &= \alpha(\mathsf{update}_{l,\mathbf{T}}(\mathsf{por}(a,b)))\quad .\end{aligned}$$

Hence, $\mathcal{R}_\alpha$ contains $\mathsf{commut}(\Sigma_{prob},\Sigma_{global})$, the commutativity laws. We conclude that α complements the tensor of equational theories.

In some cases, the equational sum coincides with the equational tensor. This is because the commutativity laws are already present in the original sets of axioms.

Example 7.5 [Nondeterminism with Error] We combine the effects of nondeterminism in Example 3.11, and error in Example 3.12. As seen in Proposition 7.1, the relation $\mathcal{R}_\alpha$ induced by the combined interpretation $\alpha = (\mathsf{Pos} * \mathsf{Err})$ contains the original axioms of the theory. One such axiom is that of idempotency, that $\mathsf{nor}(x,x)=x$. The commutativity law between nor and raise is given by $\mathsf{nor}(\mathsf{raise},\mathsf{raise})=\mathsf{raise}$, which can be proven using idempotency, substituting raise for x. Hence the combination of effects complements both the sum and the tensor of equational theories (since they are identical).

Lastly, there is an instance in which the method of combining effects neither corresponds to the equational sum, nor with the equational tensor:

Example 7.6 [Probability with Nondeterminism] We combine the effects of probability in Example 3.8, and nondeterminism in Example 3.9, using Example 6.3. The combined truth space A is given by ordered pairs (a,a') of elements $a,a'\in[0,1]$. Let $\alpha = (\mathsf{Exp} * \mathsf{Pos})$. We first show that this algebra does not complement the sum of equational theories, since the interaction law $\mathsf{por}(x,\mathsf{nor}(y,z)) = \mathsf{nor}(\mathsf{por}(x,y),\mathsf{por}(x,z))$ holds for α. Take $(a,a'),(b,b'),(c,c')\in A$:

$$\begin{aligned}\alpha(\mathsf{por}((a,a'),\mathsf{nor}((b,b'),(c,c')))) &= ((a+(b\wedge c))/2,\ (a'+(b'\vee c')/2))\\ &= ((a+b)/2\wedge(a+c)/2,\ (a'+b')/2\vee(a'+c')/2)\\ &= \alpha(\mathsf{nor}(\mathsf{por}((a,a'),(b,b')),\mathsf{por}((a,a'),(b,b'))))\quad .\end{aligned}$$

However, the commutativity law $\mathsf{por}(\mathsf{nor}(x,y),\mathsf{nor}(z,w)) = \mathsf{nor}(\mathsf{por}(x,z),\mathsf{por}(y,w))$ does not holds for α, since for $x=(0,0)$, $y=w=(1/4,1/4)$, and $z=(1,1)$:

$$\begin{aligned}\alpha(\mathsf{por}(\mathsf{nor}(x,y),\mathsf{nor}(z,w))) &= (((0\wedge 1/4)+(1\wedge 1/4))/2,\ ((0\vee 1/4)+(1\vee 1/4))/2)\\ &= (1/8,5/8)\neq(1/4,1/2)\\ &= ((0+1)/2\wedge(1/4+1/4)/2,(0+1)/2\vee(1/4+1/4)/2)\\ &= \alpha(\mathsf{nor}(\mathsf{por}(x,z),\mathsf{por}(y,w)))\quad .\end{aligned}$$

Hence, the combined interpretation α neither complements the sum, nor complements the tensor of equational theories. Instead, it complements the natural theory for this combination of effects, as e.g. described in [19,20].

8 Conclusions and related work

In this paper, we looked at Eilenberg-Moore algebras over the tree monad, whose carrier sets are given by complete lattices. These are used in [31,32] to formulate behavioural equivalence, in particular to define a *quantitative modality*: lifting quantitative predicates on values to quantitative predicates on computations. These quantitative modalities are a generalisation of Boolean modalities, given by subsets of $T^{\nu}_{\Sigma}(\{*\})$, and used in [28,29,23] to specify program equivalence. In turn, those modalities are based on the notion of *observation* from [18].

Eilenberg-Moore algebras describing weakest preconditions, as done in our example for global store, are commonly used for describing effectful programs, see e.g. [13]. In [3,21], weakest precondition semantics are given in terms of *Dijkstra monads*. It would be interesting to see whether the theory developed here has practical applications to the formalism developed in those papers.

This paper only features a selection of examples, though a lot more effects, like Input/Output, can be given an effect description. It is also possible to implement the jump effect, which is given an algebraic description in [11]. Moreover, though we only looked at a couple of combinations of effects, we can use the method developed in this paper to give a description of any combination of the effects featured in this paper.

One interesting combination we did not look at specifically is the combination of global store and error. It turns out, that this combination coincides with the tensor of equational theories. Informally, this describes the situation in which the global store is inaccessible after an error has been raised. It is however possible to define an algebra complementing the sum of equational theories, as defined in [31,32]. We can tweak interpretations of combinations of effects in other ways too. E.g., when combining probability with cost, we can associate a cost to each probabilistic choice.

Lastly, the algebras used in this paper seem suitable for defining quantitative relations, e.g. metrics [12,22], on functional programming languages with effects. For metrics, *quantales* tend to be used as space of distances. Since completely distributive lattices are quantales, there is a possible link between such metrics and the algebras used in this paper. It could be possible to develop a formalism for using algebras for defining quantitative relations, and use the method for combining effects to combine such quantitative relations. This would be an interesting avenue for future research.

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