Combining Algebraic Effect Descriptions using the Tensor of Complete Lattices

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June 2 — 6, 2020

MFPS 2020

European Union
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Introduction

Algebraic effects can alter the behaviour of functional programs.

Properties are satisfied to a certain *degree* of truth.
Such *quantitative truths* form a *complete lattice*.

To compute the degree to which a program satisfies a property, we use an *algebra* generated by a *local description*.

A truth space and description must be specified for each effect in question. This formulates several notions of *program equivalence*.

E.g.: probability, global store, nondeterminism, error, cost, I/O.

Main question: How do we generically combine such descriptions?

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Effects operations

Algebraic effects are implemented using algebraic operations.

Probabilistic operation: $\quad \mathsf{por} : X \times X \to X, \quad ar(\mathsf{por}) = 2.$

Global store operations: $\quad \mathsf{lookup}_l : X^{\mathbb{N}} \to X, \quad ar(\mathsf{lookup}_l) = \mathbb{N}.$
and for each $n \in \mathbb{N}, \quad \mathsf{update}_l(n) : X \to X, \quad ar(\mathsf{update}_l(n)) = 1.$

For each effect, a signature of operations Σ.

Effect trees

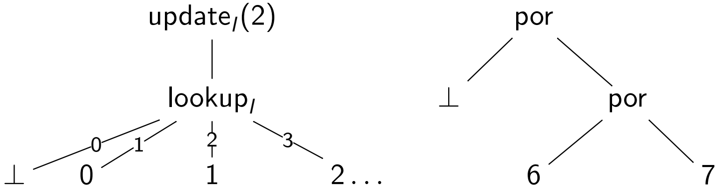
Given a signature Σ, we define a functor:

$$F_\Sigma : \quad X \quad \mapsto \quad \sum_{\mathsf{op} \in \Sigma} X^{ar(\mathsf{op})}$$

Two monads in the category of sets/posets/ω-cpos:

1. *Finite trees* $T^\mu_\Sigma(X) = \mu Y.(X \times F_\Sigma(Y))_\perp$.
2. *Infinite trees* $T^\nu_\Sigma(X) = \nu Y.(X \times F_\Sigma(Y))_\perp$.

Programs computing elements of X are denoted by trees over X.



Notions of truth

For effectful programs, the answer to:
"Does the program produce an even number?"
Is not a simple yes or no.

We need a a more nuanced notion of truth.

1. The probability that a statement is true.
2. The correct initial states for which the statement will be true.

We desire a *completely distributive complete lattice* A as truth space. This is a set A with an order $\leq$ generalising implication.

- For each $X \subset A$, there is a smallest upperbound $\bigvee X$, and a largest lower bound $\bigwedge X$ (disjunction and conjunction).
- In particular, there is a largest element $\mathbf{T} = \bigwedge \emptyset$ for always true, and a smallest element $\mathbf{F} = \bigvee \emptyset$ for never true.
- Distributivity: $\bigwedge_{i \in I} \bigvee_{j \in J_i} a_{i,j} = \bigvee_{f \in \prod_{i \in I} J_i} \bigwedge_{i \in I} a_{i,f(i)}$.

Effect Descriptions

We generate an algebra $\alpha : T^{\mu}_{\Sigma}(A) \to A$, such that we can to lift a predicate $P : X \to A$ on return values X, to a predicate $P' = (\alpha \circ T^{\mu}_{\Sigma}(P))$ on trees over X.

For each op $\in \Sigma$, we specify an *ilinear* function $\alpha_{\mathsf{op}} : A^{ar(\mathsf{op})} \to A$.

Definition

A function $f : \Pi_{i \in I} A_i \to B$ from a product of complete lattices to an complete lattice is *ilinear* if for any family of nonempty subsets $\{S_i \subseteq A_i\}_{i \in I}$, $f(\lambda i. \bigvee S_i) = \bigvee \{f(\lambda i.x_i) \mid \forall i.x_i \in S_i\}$.

The function f is *linear* if the above holds for any subsets S_i.

- This inductively generates an EM-algebra $\alpha : T^{\mu}_{\Sigma}(A) \to A$.
- In the category of ω-cpos, this can be extended to an algebra $\widehat{\alpha} : T^{\nu}_{\Sigma}(A) \to A$.

Effect Descriptions

Definition

An *effect description* (Σ, A, α) consists of an effect signature Σ, a completely distributive lattice A, and an *interpretation* α given by an ilinear function $\alpha_{\mathsf{op}} : A^{ar(\mathsf{op})} \to A$ for each op $\in \Sigma$.

For the effect of probability with signature $\Sigma = \{\mathsf{por}\}$:

- Truth space $A = [0,1]$ of probabilities of truth.
- Interpretation of expectation Exp, where
  $\mathsf{Exp}_{\mathsf{por}} : [0,1]^2 \to [0,1], \quad (a,b) \mapsto (a+b)/2.$

Global store with $\Sigma = \{\mathsf{lookup}_l, \mathsf{update}_l(n) \mid l \in L, n \in \mathbb{N}\}$:

- Truth space $A = \mathcal{P}(S)$, assertions on states $S = \mathbb{N}^L$.
- Interpretation of weakest precondition Wp, where
  $\mathsf{Wp}_{\mathsf{lookup}_l} : \mathcal{P}(S)^{\mathbb{N}} \to \mathcal{P}(S), \quad f \mapsto \{s \in S \mid s \in f(s(l))\}$
  $\mathsf{Wp}_{\mathsf{update}_l(n)} : \mathcal{P}(S) \to \mathcal{P}(S), \quad X \mapsto \{s \in S \mid s[l := n] \in X\}$

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Tensor of Complete Lattices

Definition

The *tensor product* of two complete lattices A and B is a complete lattice $A \otimes B$ such that there is a universal linear function $u_{A,B} : A \times B \to A \otimes B$ with the property that: Any linear function $f : A \times B \to C$ into a complete lattice C is the composition of $u_{A,B}$ with some linear function $f^{\otimes} : (A \otimes B) \to C$.

The *powerset representation* and the *function representation*:

1. $(A \otimes B)^{\mathcal{P}} \ := \ \{S \subseteq A \times B \ \mid \ \forall x \subseteq A, y \subseteq B, \ x \times y \subseteq S \Leftrightarrow (\bigvee x, \bigvee y) \in S\}$, with inclusion order.
2. $(A \otimes B)^{\rightarrow} \ := \ \{f : A \to B \mid \forall x \subseteq A. f(\bigvee x) = \bigwedge\{f(a) \mid a \in x\}\}$, with pointwise (extensional) order.

Examples and Properties

For a set X and a complete lattice A, $(\mathcal{P}(X) \otimes A)^{\rightarrow}$ is isomorphic to the function space $X \rightarrow A$ with pointwise order.

$$\forall f \in (\mathcal{P}(X) \otimes A)^{\rightarrow}, K \subseteq X,$$
$$f(K) = f(\bigvee_{x \in K}\{x\}) = \bigwedge_{x \in K} f(\{x\}).$$

The operation $\otimes$ is symmetric and associative.

For any complete lattice A, it holds that $(A \otimes \mathbb{B}) \simeq A$.

Proposition

The tensor product gives a symmetric monoidal product in the category of complete lattices Com, with the unit given by the Booleans $\mathbb{B}$.

Combining Functions

Universal linear map $U : A \times B \to (A \otimes B)$ given by:
$U(a, b) := \{(x, y) \in A \times B \mid x = \mathbf{F}_A \vee y = \mathbf{F}_B \vee (x \leq_A a \wedge y \leq_B b)\}$.

The linear functions of $A^I \times B^I \to C$ are precisely the linear functions of $(A \otimes B)^I \to C$, given by a map F:

$$f \quad \mapsto \quad \lambda\{S_i\}_{i \in I}. \bigvee\{f(\lambda i.a_i, \lambda i.b_i) \mid \forall i \in I.(a_i, b_i) \in S_i\}$$

Definition

For two complete lattices A, B, an indexing set I and two functions $f : A^I \to A$ and $g : B^I \to B$, we define the function $(f \otimes g) : (A \otimes B)^I \to (A \otimes B)$ as: $F(U \circ (f \times g))$.

$$(f \otimes g)(\{S_i\}_{i \in I}) = \bigvee\{U(f(\{a_i\}_{i \in I}), g(\{b_i\}_{i \in I})) \quad \mid \quad \forall i \in I.(a_i, b_i) \in S_i\}.$$

Combining Functions: Properties

Lemma

For A and B completely distributive lattices, and f and g are both ilinear, then $(f \otimes g)$ is ilinear.

The operation is associative, and has as unit conjunction over the Booleans.

Proposition

If A and B are completely distributive lattices, then $(A \otimes B)$ is a completely distributive lattice.

This follows from the observation that for $\bigwedge_A : \Pi_{i \in I} A \to A$ and $\bigwedge_B : \Pi_{i \in I} A \to A$, $(\bigwedge_A \otimes \bigwedge_B) = \bigwedge_{(A \otimes B)} : \Pi_{i \in I}(A \otimes B) \to (A \otimes B)$.

Combining Descriptions

We combine two effect descriptions $(\Sigma_1, A_1, \alpha_1)$ and $(\Sigma_2, A_2, \alpha_2)$ into an effect description $(\Sigma_{12}, A_{12}, \alpha_{12})$, where:

- $\Sigma_{12} := \Sigma_1 + \Sigma_2$,
- $A_{12} := (A_1 \otimes A_2)$,

and α_{12} is defined by the following ilinear local functions:

For op $\in \Sigma_1$, let $\alpha_{12,\mathit{inleft}(\mathsf{op})} : (A \otimes B)^{\mathit{ar}(\mathsf{op})} \to (A \otimes B)$ be $(\alpha_{1,\mathsf{op}} \otimes \bigwedge_{\mathit{ar}(\mathsf{op})})$.

For op $\in \Sigma_2$, let $\alpha_{12,\mathit{inright}(\mathsf{op})} : (A \otimes B)^{\mathit{ar}(\mathsf{op})} \to (A \otimes B)$ be $(\bigwedge_{\mathit{ar}(\mathsf{op})} \otimes \alpha_{2,\mathsf{op}})$.

This operation is associative, and has a unit give by the *pure computation description* $(\emptyset, \mathbb{B}, \downarrow)$.

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Probability + Global Store

Combined effect signature
$\Sigma = \{\mathsf{por}\} \cup \{\mathsf{lookup}_l, \mathsf{update}_l(n) \mid l \in L, n \in \mathbb{N}\}$.

Combined truth space $([0,1] \otimes \mathcal{P}(S)) \simeq (S \to [0,1])$.

Combined interpretation $\mathsf{Exp} * \mathsf{Wp}$:

$$(\mathsf{Exp} * \mathsf{Wp})_{\mathsf{por}} : (S \to [0,1])^2 \to (S \to [0,1]),$$
$$(f, g) \mapsto \lambda s.(f(s) + g(s))/2.$$

$$(\mathsf{Exp} * \mathsf{Wp})_{\mathsf{lookup}_l} : (S \to [0,1])^{\mathbb{N}} \to (S \to [0,1]),$$
$$(f_0, f_1, f_2, \dots) \mapsto \lambda s.f_{s(l)}(s).$$

$$(\mathsf{Exp} * \mathsf{Wp})_{\mathsf{update}_l(n)} : (S \to [0,1]) \to (S \to [0,1]),$$
$$f \mapsto \lambda s.f(s[l := n]).$$

Adding Nondeterminism

Effect signature $\Sigma_{non} = \{\text{nor}\}$ where $ar(\text{nor}) = 2$.

$\mathbb{A} = \{\mathbf{F}, \Diamond, \mathbf{T}\}$ where $\mathbf{F} < \Diamond < \mathbf{T}$.

Interpretation Pos: $\text{Pos}_{\text{nor}}(a, a) = a$, and $\text{Pos}_{\text{nor}}(a, b) = \Diamond$ if $a \neq b$.

We combine nondeterminism with (Σ, A, α).

$(A \otimes \mathbb{A})$ is given by the space of pairs $\{(a, b) \in A^2 \mid a \leq b\}$.

Let β be the combined interpretation given by $\alpha * \text{Pos}$.

- $\forall \text{op} \in \Sigma.\ \beta_{inleft(\text{op})}(\lambda i.(a.i, b.i)) \ = \ (\alpha_{\text{op}}(\lambda i.a_i), \alpha_{\text{op}}(\lambda i.b_i))$.
- $\beta_{inright(\text{nor})}((a_1, b_1), (a_2, b_2)) \ = \ (a_1 \wedge a_2, b_1 \vee b_2)$.

Relating to Equations

Given some countable set of variables $\mathbb{V}$, we see an element of $T^{\nu}_{\Sigma}(\mathbb{V})$ as an *algebraic expression*. A pair of algebraic expressions $(e_1, e_2) \in T^{\nu}_{\Sigma}(\mathbb{V}) \times T^{\nu}_{\Sigma}(\mathbb{V})$ expresses an *equation* $(e_1 = e_2)$.

An algebra $\alpha : T^{\nu}_{\Sigma}(A) \to A$ satisfies $(e_1 = e_2)$ if:

$$\forall f : \mathbb{V} \to A.\ \widehat{\alpha}(T^{\nu}_{\Sigma}(f)(e_1)) = \widehat{\alpha}(T^{\nu}_{\Sigma}(f)(e_2)) \ .$$

For instance, expectation $\mathrm{Exp} : T^{\nu}_{\Sigma}([0,1]) \to [0,1]$ satisfies:
$\mathrm{por}(x, x) = x$, $\mathrm{por}(x, y) = \mathrm{por}(y, x)$, $\mu x.\mathrm{por}(x, y) = y$
$\mathrm{por}(\mathrm{por}(x, y), \mathrm{por}(z, w)) = \mathrm{por}(\mathrm{por}(x, z), \mathrm{por}(y, w))$.

Combining Equational Theories

Proposition

Given two effect descriptions $(\Sigma_1, A_1, \alpha_1)$ *and* $(\Sigma_2, A_2, \alpha_2)$, *let* $(\Sigma_{12}, A_{12}, \alpha_{12})$ *be their combination. If* α_1 *satisfies* $(e_1 = e_2)$, *then* α_{12} *satisfies* $(e_1 = e_2)$.

Various options for combining effects by combining theories:

- *Sum of theories*: Only take the original equations, union of original equational theories.
- *Tensor of theories*: Take the original theories and add *commutativity laws*.

Definition

The *commutativity* law between op_1 and op_2 with arities n and m respectively, is: $\mathrm{op}_1(\lambda i.\ \mathrm{op}_2(\lambda j.\ v_{i,j})) \ = \ \mathrm{op}_2(\lambda j.\ \mathrm{op}_1(\lambda i.\ v_{i,j}))$.

Comparing Methods of Combining Effects

The combination of probability and global store behaves like the tensor of equational theories, since $\text{Exp} * \text{Wp}$ satisfies:

$$\text{por}(\text{lookup}_l(x_0, x_1, \dots), \text{lookup}_l(y_0, y_1, \dots)) =$$
$$\text{lookup}_l(\text{por}(x_0, y_0), \text{por}(x_1, y_1), \dots)$$
$$\text{por}(\text{update}_l(n)(x), \text{update}_l(n)(y)) = \text{update}_l(n)(\text{por}(x, y)).$$

Probability and Nondeterminism is neither the sum nor the tensor of equational theories, since $\text{Exp} * \text{Pos}$ satisfies:

$$\text{por}(x, \text{nor}(y, z)) = \text{nor}(\text{por}(x, y), \text{por}(x, z)).$$

but $\text{Exp} * \text{Pos}$ does not satisfy:

$$\text{por}(\text{nor}(x, w), \text{nor}(y, z)) = \text{nor}(\text{por}(x, y), \text{por}(w, z)).$$

Cost + Error

Cost effect: One operation cost_1 of arity 1, truth space $A = \{0, 1, 2, \ldots, \infty\}$ and interpretation Tal where:

$\text{Tal}_{\text{cost}_1} : A \to A, \qquad m \mapsto (m + 1).$

Tal satisfies $\text{cost}_1(\bot) = \bot$.

Error: One operation raise_e of arity 0, truth space $\mathbb{B}$ and interpretation Err where:

$\text{Err} : \mathbb{B}^0 \to \mathbb{B}, \qquad () \mapsto \mathbf{T}.$

The combination of cost and error behaves like the sum of equational theories, since $\text{Tal} * \text{Err}$ does NOT satisfy the only commutativity law:

$\text{cost}_1(\text{raise}_e) = \text{raise}_e.$

Conclusion

Quick summary:

- We have seen how to give descriptions of algebraic effects, using complete lattices and ilinear local functions. These give rise to congruent notions of program equivalence [1].
- We have seen a method of combining such descriptions using the tensor on complete lattices. This gives a natural description of combinations of effects.

Related/future research:

- Deriving descriptions directly from equational theories [2].
- Adding coeffect production to effect descriptions. Ongoing work with Tarmo Uustalu.
- Potential future work: Use effect descriptions for formulating quantitative relations, like metrics.

Thank you for listening!

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